Project – Stochastic control for market imperfection models

Forward-Backward Stochastic Differential equations and controlled McKean-Vlasov Dynamics

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Introduction – Motivation

- Recent interest in models displaying interaction between agent's state and its distribution
  - Mean Field Games (MFG)
  - Control of McKean-Vlasov system (MKV)
- In these two contexts, the control problem is non-standard : need to develop new methods and theoretical results.

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- In these two contexts, the control problem is non-standard : need to develop new methods and theoretical results.
- This article :
  - "*Forward-Backward Stochastic Differential equations and controlled McKean-Vlasov Dynamics*"
  - R. Carmona and F. Delarue
  - Develop probabilistic methods (FBSDE)
  - from the Stochastic Pontryagin Maximum Principle applied to McKean-Vlasov system.

Introduction – A non-standard problem

- Control of a specific Stochastic Differential Equations (SDE)
- SDE 'of McKen Vlasov type' (MKV-SDE hereafter) :

$$dX_t = b(t, X_t, \mathbb{P}_{X_t}, \alpha_t)dt + \sigma(t, X_t, \mathbb{P}_{X_t}, \alpha_t)dW_t$$

where

  - W : m-dim Brownian motion
  - b and σ deterministic functions
  - Controlled : *common policy* α valued in A
  - Depends on the distribution $\mathbb{P}_{X_t}$ of the *solution* of the SDE

Introduction – Control problem

- The control problem is find the optimal path $(\alpha_t)_t$.

$$J(\alpha^\star) = \inf_{\{\alpha_t\}_t} \mathbb{E}\left[\int_0^T f(t, X_t, \mathbb{P}_{X_t}, \alpha_t)dt + g(X_T, \mathbb{P}_{X_T})\right]$$

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  - Infinite dimensional differential calculus
    - Need to find the derivative of Hamiltonian *w.r.t. the measure*
    - Introduce a new formalism for $D_m H(X, \mathbb{P}_X)$

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    - Need to find the derivative of Hamiltonian *w.r.t. the measure*
    - Introduce a new formalism for $D_m H(X, \mathbb{P}_X)$
  - Common policy α :
    - Difference with MFG : here Pareto equilibrium (i.e. Social planning) compared to Nash-equilibrium
    - In MFG the control of each agent *leads* to Mean Field interaction.
    - Here : limit drawn first (mean field interaction first) and then control.

Introduction – Results and methods developed

- Stochastic Pontryagin maximum principle
  - Find an adjoint equation Y_t :
  - Will be a *backward* SDE, solved for a couple (Y_t, Z_t)
  - Necessary and sufficient conditions for optimality :
  - Maximisation of the Hamiltonian $H(\cdot, \alpha_t^\star) = \inf_\alpha H(\cdot, \alpha)$.

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- FBSDE
  - Given the (Forward)-SDE of the state X_t and the BSDE of the adjoint Y_t
  - when $\alpha_t^\star$ is the optimum of H, the system Forward-Backward will be *coupled*
  - Analysis of this system and existence/unicity result.

Introduction – Control problem

- Other results :
  - A result on the *decoupling field*
  - i.e. Expression of the adjoint Y_t as a function u of the state X_t :

$$\mathbb{P}\left(Y_t^{t,\xi} = u(t,\xi,\mathbb{P}_\xi)\right) = 1$$

  - Propagation of chaos and *approximate equilibria* :
  - Control of McKean-Vlasov dynamics provides equilibria for N players MFG with a common (i.e. exchangeable) strategy

$$\lim_{N\to\infty} \inf_{\underline{\beta}} J(\underline{\beta}) = J(\alpha^\star)$$

Preliminaries : Differentiability of function of measure

- Notion of differentiability of function with respect to measures :
  - Consider a function $H : \mathcal{P}_2(\mathbb{R}^d) \ni \mu \to H(\mu)$
  - Idea : Analyse the *lifting* (extension) $\widetilde{H}(\tilde{X})$ depending on the r.v. $\tilde{X} \in L^2$.

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- H is differentiable in μ_0 if there exists a r.v. $X_0 \sim \mu_0$ s.t. $\widetilde{H}$ is the (Fréchet) differential at $\tilde{X}_0$
  - $D\widetilde{H}(\tilde{X}_0)$ is the 'representation' of $D_\mu H(\mu_0)$
  - This derivative will be a (determ.) function $x \mapsto D_\mu H(\mu_0)(\cdot)$:

$$\begin{aligned} H(\mu) &= H(\mu_0) + D\widetilde{H}(\tilde{X}_0) \cdot (\tilde{X} - \tilde{X}_0) + o(||\tilde{X} - \tilde{X}_0||_2) \\ &= H(\mu_0) + D_\mu H(\mu_0)(\tilde{X}_0) \cdot (\tilde{X} - \tilde{X}_0) + o(||\tilde{X} - \tilde{X}_0||_2) \end{aligned}$$

Preliminaries : Differentiability w.r.t measure, an example

- Let's give a *concrete example* :
- If we define : $H(\mu) = \int_{\mathbb{R}^d} h(x)\mu(dx) = \langle h, \mu \rangle$
  - It is linear in L^2 !
- Its lifting : $\widetilde{H}(\tilde{X}) = \widetilde{\mathbb{E}}[h(\tilde{X})]$
- Its derivative : $D\widetilde{H}(\tilde{X}) \cdot Y = \widetilde{\mathbb{E}}[Dh(\tilde{X}) \cdot Y]$
- Consequently : $D_\mu H(\mu_0)(\cdot) \equiv Dh(\cdot)$
  - which is *not* equal to the Frechet-differential h.

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- Other example : function of empirical measure :
- $\bar{u}^N : \underline{x} = (x_1, x_2, \cdots, x_N) \mapsto u\big(\bar{\mu}^N\big)$
  - With $\bar{\mu}^N = \frac{1}{N} \sum_i \delta_{x_i}$
- Then $\partial_{x_i} \bar{u}^N(\underline{x}) = \frac{1}{N} D_\mu u\big(\bar{\mu}^N\big)(x_i)$

Preliminaries : other notions

- *Other notions* :
- Convergence of empirical measures $\bar{\mu}^N$:
  - In the sense of the Wasserstein distance

$$W_2(\mu,\nu) = \inf\left\{ \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, \pi(dx,dy) \Big)^{\frac{1}{2}} \;\Big|\; \pi \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \text{ with marginals } \mu \text{ and } \nu \right\}$$

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- Convergence of *functions* of empirical measure $u(\underline{x}) \to u(\mu)$ for W_2
- Convergence of the derivative : $D\bar{u}^N(\underline{x}) \to \frac{1}{N}\sum_i u(\mu)(x_i)$
  - Matters to show approximate equilibria and for the convergence of system of finitely many agents.

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  - Matters to show approximate equilibria and for the convergence of system of finitely many agents.
- Joint differentiability in (x,μ) (if the lifting is jointly diff.)
- *Convexity* : h on $\mathcal{P}_2$ – which is differentiable – is convex if :

$$h(\mu') - h(\mu) - \mathbb{E}\big[D_\mu h(\mu)(\tilde{X}) \cdot (\tilde{X}' - \tilde{X})\big] \geq 0$$

A stochastic Pontryagin Principle – Hamiltonian

- When controlling the SDE of McKean-Vlasov type, the Hamiltonian writes :

$$H(t,x,\mu,y,z,\alpha) = b(t,x,\mu,\alpha)\cdot y + \sigma(t,x,\mu,\alpha)\cdot z + f(t,x,\mu,\alpha)$$

- One can define its lifting : $H(t,x,\mathbb{P}_{\tilde{X}},y,z,\alpha) = \widetilde{H}(t,x,\tilde{X},y,z,\alpha)$
- Therefore, the derivative w.r.t. $\mathbb{P}_{\tilde{X}}$ is given by :

$$D_\mu H(t,x,\mu_0,y,z,\alpha)(\tilde{X}) = D\widetilde{H}(t,x,\tilde{X},y,z,\alpha)$$

A stochastic Pontryagin Principle – Adjoint

- Under some regularity/Lipschitzianity of coefficient (b, σ) and regularity conditions of derivatives of f and g w.r.t. x and μ, we define the (Y_t, Z_t) solution of the *adjoint* backward SDE

$$\begin{cases} dY_t = -D_x H(t, X_t, \mathbb{P}_{X_t}, \alpha_t, Y_t, Z_t)dt + Z_t dW_t - \widetilde{\mathbb{E}}\left[D_\mu H(t, \tilde{X}_t, \mathbb{P}_{X_t}, \alpha_t, \tilde{Y}_t, \tilde{Z}_t)(X_t)\right] \\ Y_T = D_x g(X_T, \mathbb{P}_{X_T}) + \widetilde{\mathbb{E}}\left[D_\mu g(\tilde{X}_T, \mathbb{P}_{X_T})(X_T)\right] \end{cases}$$

  - Tilde variables : independent copies
  - $D_\mu H(\tilde{\cdot}, \mathbb{P}_{X_t})(X_t)$: deterministic function taken in X_t.
  - Existence/uniqueness of this BSDE : provided by a suitable modification of Pardoux and Peng's proof

Pontryagin Principle – Necessary condition

- Assumption of convexity are important :
  - The Hamiltonian H is convex in α
  - The space of control A is convex
  - Regularity assumptions on the coefficients : continuity, differentiability, uniform-boundedness in initial conditions and of the derivatives, and 'at-most' linearity in (x, μ, α)
- The optimum of the control necessarily implies that the Hamiltonian is minimized :

$$H(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t, \alpha_t^\star) = \inf_{\alpha'} H(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t, \alpha') \qquad dt \otimes d\mathbb{P}\text{a.e.}$$

- The proof use perturbation arguments.

Pontryagin Principle – Necessary condition, proof

- Ideas of the proof : perturbation method :
- Variations of the objective J around the optimal $\alpha^\star$
  - Taking J at $\alpha^\varepsilon = \alpha + \varepsilon(\beta - \alpha)$
  - Computing the Gâteaux-derivative of J
    - Using notation $\theta_t = (t, X_t, \mathbb{P}_{X_t}, \alpha_t)$ and $\vartheta_T = (X_t, \mathbb{P}_{X_t})$
  - Start by defining a variation process V_t, being the 'First-order approximation' [*Lem. 4.1*] of the perturbed process : $X^{\alpha^\varepsilon} =: X^\varepsilon$

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{0 \le t \le T} \left| \frac{X^\varepsilon - X}{\varepsilon} - V_t \right|^2\right] = 0$$

  - V_t is complicated and is composed of derivatives of (b, σ) w.r.t. the variables (x, μ, α).
  - Computing the G-diff of J [*Lem. 4.2, Cor. 4.4*] in α :

$$\lim_{\varepsilon \to 0} \frac{d}{d\,\varepsilon} J(\alpha^\varepsilon) = \mathbb{E}\int_0^T \left[D_x f(\theta_t) \cdot V_t + \widetilde{\mathbb{E}}(D_\mu f(\theta_t)(\tilde{X}_t)) + D_\alpha f(\theta_t) \cdot (\beta_t - \alpha_t)\right] dt$$

$$+ \mathbb{E}\left[D_x g(\vartheta_T) \cdot V_T + \widetilde{\mathbb{E}}(D_\mu g(\vartheta_T)(\tilde{X}_t) \cdot \tilde{V}_T)\right]$$

Pontryagin Principle – Necessary condition, proof

- Ideas of the proof : perturbation method :
- Variations of the objective J around the optimal $\alpha^\star$
  - Using the reexpression of the last term [*Lem. 4.3 &Cor. 4.4*], as an integral over time, one can obtain :

$$\lim_{\varepsilon \to 0} \frac{d}{d\,\varepsilon} J(\alpha^\varepsilon) = \mathbb{E} \int_0^T \left[D_\alpha H(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t, \alpha_t) \cdot (\beta_t - \alpha_t) \right] dt$$

  - All these, obtained through chain-rule argument, but this time with function of measures.
  - By optimality condition :

$$\lim_{\varepsilon \to 0} \frac{d}{d\,\varepsilon} J(\alpha + \varepsilon(\beta - \alpha)) \geq 0$$

  - Thanks to convexity of the Hamiltonian in α, obtain :

$$H(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t, \beta_t) \geq H(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t, \alpha_t) \qquad dt \otimes d\mathbb{P}\text{a.e.}$$

Pontryagin Principle – Sufficient condition

- Under convexity assumptions :
  - Convexity of the cost function : $(x,\mu) \mapsto g(x,\mu)$ and $(x,\mu,\alpha) \mapsto H(t,x,\mu,Y_t,Z_t,\alpha) \quad dt \otimes d\mathbb{P}$a.e.
  - The same regularity assumptions as above
  - If X is solution of the McKean-Vlasov SDE and $(Y_t,Z_t)_{t\in[0,T]}$ the adjoint processes,
- If :

$$H(t,X_t,\mathbb{P}_{X_t},Y_t,Z_t,\alpha_t^\star) = \inf_{\alpha'} H(t,X_t,\mathbb{P}_{X_t},Y_t,Z_t,\alpha') \qquad dt \otimes d\mathbb{P}-a.e.$$

- It is sufficient for the optimal control $J(\alpha^\star) = \inf_{\alpha'} J(\alpha')$.

Pontryagin Principle – Sufficient condition, proof

- Ideas of the proof : relies deeply on the assumption of convexity
- Express the difference $J(\alpha^\star) - J(\alpha')$
  - In terms of differences in the functions g and f
  - Using $f(\theta_t) = H(\theta_t) - (b \cdot y + \sigma)(\theta_t)$
  - Use the convexity of H w.r.t. (x, μ)

$$J(\alpha^\star)-J(\alpha') \leq \mathbb{E}\int_0^T \Big\{H(\theta_t)-H(\theta_t')-D_xH(\theta_t)\cdot(X_t-X_t')+\widetilde{\mathbb{E}}\big[D_\mu H(\widetilde{\theta})(X_t)\cdot(\tilde{X}_t-\tilde{X}_t)\big]\Big\}dt$$

- implying the optimality of $\alpha^\star$

Reformulation as a Forward-Backward SDE

- Now, we had a SDE for X_t and a BSDE for (Y_t, Z_t)
- Idea : with a sufficient and necessary condition, need to use probabilistic methods to solve the control.
- Obtain a FBSDE system that is coupled by the optimal control :

$$\alpha^\star(t, X_t, \mathbb{P}_X, Y_t, Z_t) \in \underset{\alpha}{\operatorname{argmin}}\, H(\cdot, \alpha)$$

- Require to restrict the model, i.e. with assumptions :
  - (i) the coefficients linear in the first-moment of the law of state $\bar{\mu} = \int x d\mu(x)$
  - (ii) the above regularity assumptions on functions/coefficients
  - (iii) Lipschitz-continuity of Df, Dg (w.r.t. x, μ or α)
  - (iv) Convexity of f and thus H in (x, μ, α)

Reformulation as a Forward-Backward SDE

- The FBSDE is reformulated :

$$\begin{aligned} dX_t &= \Big[b_t^0 + b_t^1\mathbb{E}[X_t] + b_t^2 X_t + b_t^3\alpha^\star(t, X_t, \mathbb{P}_X, Y_t, Z_t)\Big]dt + \\ &\qquad \big[\sigma_t^0 + \sigma_t^1\mathbb{E}[X_t] + \sigma_t^2 X_t + \sigma_t^3\,\alpha^\star(t, X_t, \mathbb{P}_X, Y_t, Z_t)\big]dW_t \qquad (1) \\ dY_t &= -\Big[D_x f\big(t, X_t, \mathbb{P}_{X_t}, \alpha^\star(t, X_t, \mathbb{P}_X, Y_t, Z_t)\big) + b_t^2 Y_t + \sigma_t^2 Z_t\Big]dt + Z_t dW_t \\ &\qquad - \Big\{\widetilde{\mathbb{E}}\big[D_\mu f(t, X_t, \mathbb{P}_X, \alpha^\star(t, \tilde{X}_t, \mathbb{P}_X, \tilde{Y}_t, \tilde{Z}_t))(X_t)\big] + b_t^1\mathbb{E}[Y_t] + \sigma_t^1\mathbb{E}[Z_t]\Big\}dt \\ X_0 &= x_0 \qquad Y_T = D_x g(X_T, \mathbb{P}_{X_T}) + \widetilde{\mathbb{E}}\big[D_\mu g(\tilde{X}_T, \mathbb{P}_{X_T})(X_T)\big] \end{aligned}$$

where $b_t^0, b_t^1, b_t^2, b_t^3, \sigma_t^0, \sigma_t^1, \sigma_t^2$ and σ_t^3 are the parameters of the model

- Under the above conditions, this FBSDE *has a unique solution*.

Forward-Backward SDE – Existence and unicity

- Based on the continuation method :
  - Use the result of existence and uniqueness when the FBSDE is *known* to hold.
  - And show the result is preserved when the coefficients are perturbed.
  - Linear perturbations, (natural), which justify the restriction of the model.
    - To insure that the function $(t, x, \mu, y, z) \mapsto \alpha^\star(t, x, \mu, y, z)$ is locally bounded and Lipschitz continuous w.r.t. (x, μ, y, z)
    - In particular, the Lipschitz-property w.r.t. μ is non-standard and is proved by the use of (iii), convexity of f and $\alpha^\star$ being critical point of the Hamiltonian.

Forward-Backward SDE – Existence and unicity, proof

- Ideas of the proof : Continuation method :
- Reformulation of the FBSDE :
  - Discounting (b, σ) and $(D_x H, D_\mu H)$ and $(D_x g, D_\mu g)$ by γ,
  - Adding linearly a perturbation $\mathcal{I} = (\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f, \mathcal{I}^g)$
    - Using the notation $\Theta_t = (t, X_t, \mathbb{P}_{X_t}, \alpha_t, Y_t, Z_t)$, $\theta_t = (t, X_t, \mathbb{P}_{X_t}, \alpha_t)$ and $\vartheta_T = (X_T, \mathbb{P}_{X_T})$:

$$\begin{cases} dX_t = \left(\gamma b(\theta_t) + \mathcal{I}_t^b\right)dt + \left(\gamma\sigma(\theta_t) + \mathcal{I}_t^\sigma\right)dW_t \\ dY_t = -\left(\gamma D_x H(\Theta_t) + \widetilde{\mathbb{E}}\left[D_\mu H(\tilde{\Theta}_t)(X_t)\right] + \mathcal{I}_t^f\right)dt + Z_t dW_t \\ Y_T = \gamma\left(D_x g(\vartheta_T) + \widetilde{\mathbb{E}}\left[D_\mu g(\tilde{\vartheta}_T)(X_T)\right]\right) + \mathcal{I}_T^g \\ \alpha_t = \alpha^\star(\Theta_t) \end{cases}$$

  - This formulation is $\mathcal{F}(\gamma, \xi, \mathcal{I})$ for initial condition $X_0 = \xi$
  - Property $\mathcal{S}_\gamma$ holds true when the FBSDE $\mathcal{F}(\gamma, \xi, \mathcal{I})$ – for any $\xi \in L^2$ and any $\mathcal{I}$ – has a unique solution.

Forward-Backward SDE – Existence and unicity, proof

- Ideas of the proof :
- Based on [*Lemma 5.4*] : if there exists a $\gamma \in [0, 1)$ s.t. $\mathcal{S}_\gamma$ holds true, then there exists δ_0 s.t. $\mathcal{S}_{\gamma+\eta}$ holds true for $\eta \leq \delta_0$ and $\gamma + \eta \leq 1$.
- The proof of this lemma is based on Picard's contraction theorem.

- The existence and uniqueness is thus proved :
  - Since $\mathcal{S}_0$ – the trivial solution of the FBSDE – is known
  - Induction on η up to $\mathcal{S}_1$ and $\mathcal{I} \equiv 0 \Rightarrow$ prove the result for the FBSDE eq. (1).

Other results – Decoupling field

- Main difficulty of this FBSDE is that X_t and Y_t are coupled by the optimum $\alpha^\star(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t)$
- There exists a *decoupling field* :
  - Measurable mapping from the solution of the SDE to the solution of the BSDE
- Holds for a specific initial value : ξ

$$Y_t^\xi = u(t, \xi, \mathbb{P}_\xi) \qquad \text{a.e.}$$

- Holds for the whole space :

$$\forall t \in [0, T] Y_t^\xi = u(t, X_t^\xi, \mathbb{P}_\xi) \qquad \text{a.e.}$$

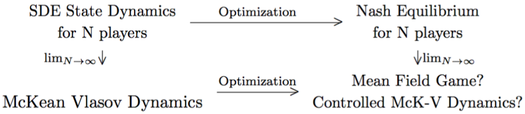
- The function will satisfy the master equation PDE.
- Open question : difficulty when the coefficient (b, σ, f, g) depends on randomness (+ common noise).

Difference with MFGs

- Difference between Mean Field Games and Control of McKean Vlasov
- Reference : Carmona, Delarue and Lachapelle (2013),
- Both models : asymptotic behavior of stochastic differential games when the number of players goes to infinity.
- Which notion of equilibrium we consider :
  - MFG : Nash-equilibrium,
    - When an agent optimize, considers the worst possible outcome of the other players
    - The measure of agents is *fixed*
  - Control of McKean-Vlasov : Pareto (cooperative) optimum :
    - When the social-planner optimize, it *does* change the distribution
    - Requires the derivative of the Hamiltonian w.r.t. α and w.r.t. μ.

Difference with MFGs – Probabilistic approach

- Difference between Mean Field Games and Control of McKean Vlasov
- Reference : Carmona, Delarue and Lachapelle (2013),
  - Question : order in which one perform the optimization (control) and the passage to the limit :



- Resolution using probabilistic approach : Pontryagin principle and FBSDE (coupled !)
- But not of McKean-Vlasov type : no *change* of $\mathbb{P}_X$ when perturbing α.

HJB on the space of measure

- Idea : In this setting, the SDE of McKean Vlasov setting is non-Markovian
- Expand the state space : from $\mathbb{R}^d$ to $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$
- Flow property :
  - If μ is the (initial) law of X_0
  - Defining $\mathbb{P}_{X_t^{t_0,x_0}} =: \mathbb{P}_t^{t_0,\mu_0}$
  - This will imply $\mathbb{P}_t^{t_0,\mu_0} = \mathbb{P}_t^{s,\mathbb{P}_s^{t_0,\mu_0}}$.
- $\Rightarrow$ restore the Markov property of the state process.

HJB on the space of measure

- Restart from the control problem :

$$v(t_0,\mu_0) = \inf_{\{\alpha_t\}_{t_0}^T} \mathbb{E}\left[\int_{t_0}^T f(t, X_t, \mathbb{P}_t^{t_0,\mu_0}, \alpha)dt + g(X_T, \mathbb{P}_{X_T})\right]$$

- Define $f^{\mathbb{E}}(t,\mu,\alpha) := \langle f\big(t,\cdot,\mu,\alpha(t,\cdot,\mu)\big),\mu\rangle = \hat{\mathbb{E}}^{\mu}\Big(f(t,\hat{X}_t,\mu,\alpha(t,\hat{X}_t,\mu))\Big)$, and $g^{\mathbb{E}}(\mu) = \langle g(\cdot,\mu),\mu\rangle$.
- With Fubini's theorem and

$$v(t_0,\mu_0) = \inf_{\alpha}\left[\int_{t_0}^T f^{\mathbb{E}}\big(t,\mathbb{P}_t^{t_0,\mu_0},\alpha(\cdot)\big)dt + g^{\mathbb{E}}(\mathbb{P}_T^{t_0,\mu})\right]$$

- Yields the DPP [*Thm 3.1*] :

$$v(t_0,\mu_0) = \inf_{\alpha}\left[\int_{t_0}^{\tau} f^{\mathbb{E}}\big(t,\mathbb{P}_t^{t_0,\mu_0},\alpha(\cdot)\big)dt + v(\tau,\mathbb{P}_{\tau}^{t_0,\mu_0})\right]$$

HJB on the space of measure

- Using the same notion of differentiability w.r.t. measure as above, one can prove the corresponding Itô's formula.
- Obtain the infinitesimal generator :

$$\mathcal{L}_t^{\alpha} v(t,\mu)(x) = D_\mu v(t,\mu)(x) \cdot b(t,x,\mu,\alpha(t,x,\mu)) + \frac{1}{2}\operatorname{Tr}\left(D_x D_\mu v(t,\mu)(x)\sigma\sigma^T(t,x,\mu,\alpha(t,x,\mu))\right)$$

- the HJB is the following :

$$\partial_t v + \inf_\alpha \left[f^{\mathbb{E}}(t,\mu,\alpha) + \langle \mathcal{L}_t^{\alpha} v(t,\mu)(\cdot), \mu \rangle \right] = 0 \qquad \text{on} \quad [0,T) \times \mathcal{P}_2(\mathbb{R}^d)$$

$$v(T,\cdot) = g^{\mathbb{E}} \qquad \text{on} \quad \mathcal{P}_2(\mathbb{R}^d)$$

- 'Standard' verification methods :
  - Supposing w is bounded in $\mathcal{C}^{1,2}\left([0,T] \times \mathcal{P}_2(\mathbb{R}^d)\right)$
  - and solution of HJB and $\alpha^\star$ realize the inf. of the Hamiltonian
  - then $w = v$ and the optimal control is given in feedback form by $\alpha^\star$.
- Viscosity solution :
  - The value function (defined on the space of measure !) is a viscosity solution to the HJB [*Prop 5.1*].

Discussion and conclusion

- Complete, exhaustive article
- Several restrictive assumptions for the proof of existence/uniqueness of the coupled FBSDE.
- Several results not so new (Differentiability w.r.t. measure, Pontryagin Principle for MKV SDE)
- However, very interesting subject, pedagogical approach, link with many other theories.

- *Thank you for you attention !*

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